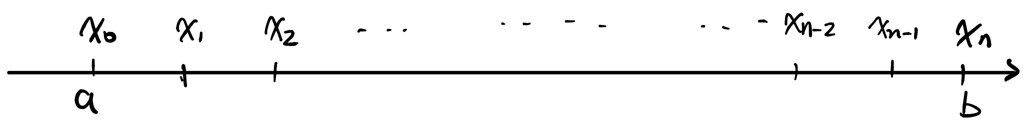
Riemman Integration

Def: $a, b \in \mathbb{R}$, $a<b$. A partision of the closed interval $[a,b]$ is $a = x_0 < x_1 < \cdots < x_n = b$.



the width of the partision is $\max\limits_{1 \le i \le n} (x_i - x_{i-1})$.

Def: $f : [a, b] \mapsto \mathbb{R}$. A Reimman sum of f is

$$S = \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1})$$

where $\alpha_i \in [x_{i-1}, x_i]$.

Def: $f : [a,b] \mapsto \mathbb{R}$ is Reimman integrable if $\exists I \in \mathbb{R}$, s.t. $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t.

$|S - I| < \varepsilon$ for all Reiman sums with width less than δ $\left(\max\limits_{1 \le i \le n} |x_i - x_{i-1}| < \delta \right)$.

In the case, I is called the Reimman Integral of f

between a and b, denoted as $I = \int_a^b f(x)\,dx$.

Example

(1) $f : [a, b] \mapsto \mathbb{R}$, $f(x) = C$ a constant for all x.

$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a partition of $[a, b]$.

$\alpha_i \in [x_{i-1}, x_i]$.

$$S = \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) = C \sum_{i=1}^{n} (x_i - x_{i-1}) = C(b-a).$$

(2) $f : [0, 1] \mapsto \mathbb{R}$, $f(x) = x$

$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, $\alpha_i \in [x_{i-1}, x_i]$

$S = \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1})$, width $\delta = \max_{1 \le i \le n} \{x_i - x_{i-1}\}$.

so, $x_{i-1} < \alpha_i < x_i$. On the other hand

$$x_{i-1} < \frac{x_i + x_{i-1}}{2} < x_i$$

so, $$\left| \alpha_i - \frac{x_i + x_{i-1}}{2} \right| < \max_{1 \le i \le n} |x_i - x_{i-1}| = \delta$$

Since $$I = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1})$$

$$= \frac{1}{2} \sum_{i=1}^{n} x_i^2 - x_{i-1}^2 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}$$

Let $\varepsilon > 0$, select $\delta = \varepsilon$.

Thus, $|S - I| = \left| \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) - I \right|$

$$= \left| \sum_{i=1}^{n} \alpha_i (x_i - x_{i-1}) - \sum_{i=1}^{n} \frac{x_i + x_{i-1}}{2} (x_i - x_{i-1}) \right|$$

$$\leq \sum_{i=1}^{n} \left| \alpha_i - \frac{x_i + x_{i-1}}{2} \right| (x_i - x_{i-1})$$

$$< \delta \sum_{i=1}^{n} (x_i - x_{i-1}) = \delta = \varepsilon.$$

Thus, $\int_0^1 f(x)\,dx = I = \frac{1}{2}$.

<u>Properties</u> $f : [a,b] \mapsto \mathbb{R}$, $g : [a,b] \mapsto \mathbb{R}$

$c \in \mathbb{R}$, f, g are Reimman integrable.

Prove :

(1) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(2) $\int_a^b c f(x)\,dx = c \int_a^b f(x)\,dx$.

i.e. integral operator $\int_a^b \cdot\,dx$ is linear.

<u>proof</u> :

(1) Let $\varepsilon > 0$, since f and g are Reimman integrable

$\exists\, \delta_1, \delta_2 > 0$, s.t. If S_f is a Reimman sum of f with partition width $< \delta_1$, and S_g is a Reimman sum of g with partition width $< \delta_2$.

$$\left| S_f - \int_a^b f(x)\,dx \right| < \varepsilon/2$$

$$\left| S_g - \int_a^b g(x)\,dx \right| < \varepsilon/2$$

Since the Reimman sum $S_{f+g} = \sum_{i=1}^{n} (f+g)(\alpha_i)(x_i - x_{i-1})$ $= \sum_{i=1}^{n} f(\alpha_i)(x_i - x_{i-1}) + \sum_{i=1}^{n} g(\alpha_i)(x_i - x_{i-1}) = S_f + S_g$.

Select $\delta = \min\{\delta_1, \delta_2\}$. If S_{f+g} is a Reimman sum with partition width $< \delta$, then

$$\left| S_{f+g} - \left(\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right)\right| = \left|\left(S_f - \int_a^b f(x)\,dx\right) + \left(S_g - \int_a^b g(x)\,dx\right)\right| \le \left| S_f - \int_a^b f(x)\,dx\right| + \left| S_g - \int_a^b g(x)\,dx\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(2) is similar to (1).

Proposition

$f : [a,b] \mapsto \mathbb{R}$ integrable. $f(x) \geqslant 0 \ \forall x \in [a,b]$.

then $\int_a^b f(x)dx \geqslant 0$.

Corollary: $f [a,b] \mapsto \mathbb{R}$ integrable, $f(x) \geqslant g(x), \forall x \in [a,b]$.

then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

proof: consider $\varphi(x) = f(x) - g(x) \geqslant 0, \ \forall x \in [a,b]$.

then $\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b \varphi(x)dx \geqslant 0$.

so. $\int_a^b f(x)dx \geqslant \int_a^b g(x)\,dx$.

Corollary $f : [a,b] \mapsto \mathbb{R}$ integrable, $m \leq f(x) \leq M, \forall x \in [a,b]$.

then $m(b-a) \leqslant \int_a^b f(x)dx \leq M(b-a)$.

Lemma $f : [a,b] \mapsto \mathbb{R}$, f is integrable $\Leftrightarrow \forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $|S_1 - S_2| < \varepsilon$, if S_1 and S_2 are two Reiman sums with width $< \delta$.

~~proof~~ :

$\Rightarrow$) Let $\varepsilon > 0$, since f is integrable, $\exists\, \delta > 0$ s.t. $\left| S - \int_a^b f(x)\,dx \right| < \varepsilon/2$, if S is a Reiman sum with width $< \delta$.

Let S_1 and S_2 are Reiman sums of f with partition width $< \delta$.

Then $|S_1 - S_2| \leq \left|S_1 - \int_a^b f(x)\,dx\right| + \left|\int_a^b f(x)\,dx - S_2\right|$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\Leftarrow$) Let $\varepsilon = \frac{1}{n}$, $\exists\, \delta_n > 0$ s.t.

$|S_n^{(1)} - S_n^{(2)}| < \frac{1}{n}$ if $S_n^{(1)}$ and $S_n^{(2)}$ are two Reimman sums with partition width $< \delta_n$.

Now, we assume $\delta_{n+1} \leq \delta_n$.

Select $N = \frac{1}{\varepsilon}$, if $n, m > N$, then

$|S_n - S_m| < \frac{1}{\min\{n,m\}} \leq \frac{1}{N} < \varepsilon$.

this shows that S_n is Cauchy, then $S_n \to I$.

Let $\varepsilon > 0$, $N > \frac{1}{\varepsilon}$, if S is a Reimman

with partition width $< \frac{1}{N} \Rightarrow |S - I| \leq |S - S_m|$
$+ |S_m - I| < \varepsilon + 0 = \varepsilon.$